MACROSCOPIC NON-UNIQUENESS AND LIMITS OF HAMILTONIAN DYNAMICS

S. DOSTOGLOU AND JIANFEI XUE

Abstract. We construct explicit examples of spontaneous energy generation and non-uniqueness for the compressible Euler system, with and without pressure, by taking limits of Hamiltonian dynamics as the number of molecules increases to infinity. The examples come from rescalings of well-posed, deterministic systems of molecules that either collide elastically or interact via singular pair potentials.

Contents

Date: September 29, 2016.

2000 *Mathematics Subject Classification.* 76, 82C,28, 35, 37J,K.

1. Introduction

Non-uniqueness for weak solutions of hydrodynamic equations is well known. Examples include the construction of V. Scheffer [Sch] and A. Shnirelman [Sh] of non-trivial weak solutions of (incompressible, two-dimensional) Euler equations with compact time and space support, and the work by C. De Lellis and L. Székelyhidi [dLS] showing that non-uniqueness (of the incompressible and compressible Euler equations in dimension greater or equal to two) persists even under "admissibility" conditions. [D] is a standard reference on the non-uniqueness of weak solutions of hyperbolic conservation laws in general.

In an attempt to investigate the origin of this behavior, we adopt here the point of view that hydrodynamic equations are the result of averaging microscopic evolution equations (cf. [M], p. 81, and [B], Part I, §20) to construct explicit examples of spontaneous macroscopic energy generation and non-uniqueness for the compressible Euler system, with and without pressure, as limits of Hamiltonian dynamics. Our examples are rescaled limits of well-posed, deterministic systems of molecules that either collide elastically or interact via rescaled, singular pair interaction potentials, at the limit of infinitely many molecules, cf. C.B. Morrey's work [Mor]. For each moment t and finite N, the positions and velocities of the

molecules define the probability measure $M_t^{(N)}(d\mathbf{x}, d\mathbf{v}) := \dfrac{1}{N}\sum_{k=1}^{N}\delta_{(\mathbf{x}_k,\mathbf{v}_k)}(d\mathbf{x}, d\mathbf{v})$. In all examples here, $M_t^{(N)}$ converges weakly to M_t as $N \to \infty$ and for each $(t, \mathbf{x})$ the macroscopic density is given by the first marginal of M_t and the macroscopic velocity by the barycentric projection of M_t at $\mathbf{x}$ with respect to this marginal.

The first part of this article, consisting of Sections 3 and 4, is centered on an example showing spontaneous generation of macroscopic velocitiy. The microscopic systems start with groups of motionless molecules and a single molecule, macroscopically undetectable, initially at a sufficiently large distance from the group, moving towards the group. Macroscopically, the limit of these flows describes a line segment in $\mathbb{R}^2$ at rest for $t \in (-\infty, 0]$, which splits into two equal parts moving away from each other with velocities ± 1 as soon as t becomes positive. The macroscopic velocity and the macroscopic density from M_t turn out to be a weak solution of the 2-dimensional presureless Euler for all t in $\mathbb{R}$. This solution is macroscopically as “inadmissible” as those of Scheffer and Shnirelman in that kinetic energy is spontaneously created at t_0. (Microscopically, total energy is, of course, conserved.) As Hamiltonian flows are time reversible, in Section 4 the flows $M_t^{(N)}$ are reversed to produce a solution to the 2-dimensional presureless Euler that does decrease energy. When compared to an elementary transverse flow, this provides an example of non-uniqueness for the presureless Euler under the admissibility condition of non-increasing energy.

In the second part of this article, Section 5 provides an interpretation, via a microscopic derivation, of the well known non-uniqueness of the Cauchy problem for the 1-dimensional Euler system. We show how three moment equations derived from the transport equation

$$\partial_t M_t + v\partial_x M_t = 0 \tag{1.1}$$

can result in the 1-dimensional Euler system. The main point here is that two flows of probability measures solving the same transport equation, even if their moments coincide at $t = 0$, in general will not have identical moments for all

later times. Indeed, we construct two limit measures M_t and $\widetilde{M_t}$ both solving (1.1) and resulting in the 1-dimensional Euler system. At $t = 0$, both M_t and $\widetilde{M_t}$ give the same macroscopic density, velocity, and pressure. Macroscopically, the solutions produced by M_t and $\widetilde{M_t}$ can be pictured as a segment of two and three layers, respectively, on top of each other moving freely, see Figures 11 and 13. The solutions in Section 5 are surrounded by vacuum (zero density).

2. Preliminaries and Notation

2.1. **Measure theory.** Recall that a sequence of finite measures $M_n(dx)$ converges *weakly* to a finite measure $M(dx)$ if for any $f(x)$ continuous and bounded $\int f(x)M_n(dx) \to \int f(x)M(dx)$, $n \to \infty$. We then write $M_n \Rightarrow M$.

For $f : X \to Y$ measurable and M a probability measure on X the *push-forward measure* $f_{\#}M$ of f on Y (the distribution measure of the random variable f) is $(f_{\#}M)(B) = M(f^{-1}(B))$. We often write fM for this push-forward.

If M is on $\mathbb{R}^{2d}$ its *first marginal* will be $(\pi_1)_{\#}M$, for $\pi_1 : \mathbb{R}^{2d} \to \mathbb{R}^d, \pi_1(\mathbf{x}, \mathbf{v}) = \mathbf{x}$.

$\int M_{\mathbf{x}}(d\mathbf{v})\mu(d\mathbf{x})$ is a shorthand for the measure $f \mapsto \int \left(\int f(\mathbf{x}, \mathbf{v}) M_{\mathbf{x}}(d\mathbf{v}) \right) \mu(d\mathbf{x})$.

The *disintegration* of $M(d\mathbf{x}, d\mathbf{v})$ with respect to its first marginal $\mu(d\mathbf{x})$ is the unique, up to a μ-measure 0, family $M_{\mathbf{x}}(d\mathbf{v})$ such that $M(d\mathbf{x}, d\mathbf{v}) = \int M_{\mathbf{x}}(d\mathbf{v})\mu(d\mathbf{x})$. The barycentric projection of this disintegration is $\overline{\mathbf{v}}(\mathbf{x}) = \int \mathbf{v} M_{\mathbf{x}}(d\mathbf{v})$ for $\mathbf{x}$ in the support of μ, $\overline{\mathbf{v}} = 0$ otherwise. For details see [AGS], Section 5.3, or [DJX], Section 3.1.

2.2. **Finite systems.** A system of N molecules in $\mathbb{R}^d$ will be described by the positions and velocities of the molecules, $(\mathbf{x}_k(t), \mathbf{u}_k(t))$, $1 \le k \le N$, evolving via

Hamiltonian dynamics with pairwise interaction $\Phi_\sigma(r)$ of finite range σ:

$$\begin{aligned}
&\frac{d}{dt}\mathbf{x}_k(t) = \mathbf{u}_k(t),\\
&\frac{d}{dt}\mathbf{u}_k(t) = -\frac{1}{N}\sum_{\substack{j=1\\ j\neq k}}^{N} \Phi'_\sigma\left(|\mathbf{x}_k(t)-\mathbf{x}_j(t)|\right)\frac{\mathbf{x}_k(t)-\mathbf{x}_j(t)}{|\mathbf{x}_k(t)-\mathbf{x}_j(t)|}.
\end{aligned} \tag{2.1}$$

Following Morrey [Mor], we shall take $\Phi_\sigma(r) = \Phi\left(\frac{r}{\sigma}\right)$ for some $\Phi : (0,\infty) \to [0,\infty)$ satisfying:

$$\lim_{r\to 0}\Phi(r) = +\infty, \quad \Phi' \leq 0, \quad \Phi'' \geq 0, \quad \Phi(r) \neq 0 \Leftrightarrow 0 < r < 1. \tag{2.2}$$

For each N, suppose that a system $\left(\mathbf{x}_k^{(N)}(t), \mathbf{u}_k^{(N)}(t)\right)$, $k = 1,\ldots,N$ evolves according to (2.1). Of central importance will be the corresponding t-family of probability measures on $\mathbb{R}^{2d}$:

$$M_t^{(N)}(d\mathbf{x}, d\mathbf{v}) := \frac{1}{N}\sum_{k=1}^{N}\delta_{\left(\mathbf{x}_k^{(N)}(t),\mathbf{u}_k^{(N)}(t)\right)}(d\mathbf{x}, d\mathbf{v}),\ \ t \geq 0, \text{ or } t \in \mathbb{R}. \tag{2.3}$$

When $M_t^{(N)}$ converges weakly to some M_t, it is crucial to note that the empirical measure formed by neglecting a single molecule converges weakly to the same M_t. (In fact, neglecting $o(N)$ number of molecules has the same effect.) In this sense, any single molecule is macroscopically *invisible*. The construction in Section 3 relies heavily on this observation.

3. Spontaneous Macroscopic Velocity Generation from Hamiltonian Dynamics

This section presents an example of a microscopic Hamiltonian flow with macroscopic limit, as $N \to \infty$, that shows spontaneous velocity generation. The microscopic systems start with groups of motionless molecules and a single molecule, initially at a sufficiently large distance from the group, moving towards the group with large velocity. For $t < 0$, as $N \to \infty$, the moving molecule is invisible and the macroscopic system is motionless. However, as the moving molecule starts interacting with the group at $t = 0$, its energy is transferred to the rest of the system in

such a way that all other molecules acquire speed 1 to create macroscopic velocity for $t > 0$.

There are similarities here with Lanford [L], pp. 50–53, although Lanford works with an infinite system of hard balls that always remains discrete, rather than the limit of finite Hamiltonian systems with interaction, and he does not obtain hydrodynamic equations.

Throughout this section we use $\mathbf{Q}_t$ for the segment

$$\{(x, y) : 0 \leq x \leq 1, y = t\} \subset \mathbb{R}^2 \tag{3.1}$$

and $\Delta_t(d\mathbf{x})$ for the normalized 1-dimensional Lebesgue measure on $\mathbf{Q}_t$.

Theorem 3.1. *For each $N \in \mathbb{N}$, there exists $\sigma_N > 0$ and $\left(\mathbf{x}_k^{(N)}(t), \mathbf{u}_k^{(N)}(t)\right)$, $k = 1, \ldots, N$, solution of the Hamiltonian system* (2.1) *with interaction Φ_{σ_N} for all $t \in \mathbb{R}$, such that for all $t \in \mathbb{R}$, the sequence of empirical measures*

$$M_t^{(N)}(d\mathbf{x}, d\mathbf{v}) := \frac{1}{N} \sum_{k=1}^{N} \delta_{\left(\mathbf{x}_k^{(N)}(t), \mathbf{u}_k^{(N)}(t)\right)}(d\mathbf{x}, d\mathbf{v}) \tag{3.2}$$

converges weakly, as $N \to \infty$, to

$$M_t(d\mathbf{x}, d\mathbf{v}) = \begin{cases} \Delta_0(d\mathbf{x}) \otimes \delta_{(0,0)}(d\mathbf{v}) & t \leq 0 \\ \frac{1}{2}\Delta_t(d\mathbf{x}) \otimes \delta_{(0,1)}(d\mathbf{v}) + \frac{1}{2}\Delta_{-t}(d\mathbf{x}) \otimes \delta_{(0,-1)}(d\mathbf{v}) & t > 0. \end{cases} \tag{3.3}$$

The proof of this theorem occupies the rest of this section. For the moment, note that the first marginal (macroscopic density) of $M_t(d\mathbf{x}, d\mathbf{v})$ in (3.3) is

$$\mu_t(d\mathbf{x}) = \begin{cases} \Delta_0(d\mathbf{x}) & t \leq 0 \\ \frac{1}{2}\Delta_t(d\mathbf{x}) + \frac{1}{2}\Delta_{-t}(d\mathbf{x}) & t > 0, \end{cases} \tag{3.4}$$

If we disintegrate

$$M_t(d\mathbf{x}, d\mathbf{v}) = \int M_{t,\mathbf{x}}(d\mathbf{v})\, \mu_t(d\mathbf{x}) \tag{3.5}$$

then

$$M_{t,\mathbf{x}}(d\mathbf{v}) = \begin{cases} \chi_{\mathbf{Q}_0}(\mathbf{x})\delta_{(0,0)}(d\mathbf{v}) & t \leq 0 \\ \chi_{\mathbf{Q}_t}(\mathbf{x})\delta_{(0,1)}(d\mathbf{v}) + \chi_{\mathbf{Q}_{-t}}(\mathbf{x})\delta_{(0,-1)}(d\mathbf{v}) & t > 0. \end{cases} \tag{3.6}$$

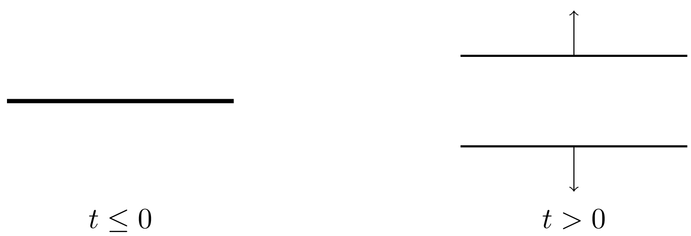

FIGURE 1. Macroscopic flow of M_t in Theorem 3.1.

Notice that in (3.6) we only needed to specify $M_{t,\mathbf{x}}(d\mathbf{v})$ for $\mathbf{x}$ in the support of $\mu_t(d\mathbf{x})$. The macroscopic velocity is the barycentric projection of this disintegration:

$$\mathbf{u}(t,\mathbf{x}) := \int_{\mathbb{R}^2} \mathbf{v} M_{t,\mathbf{x}}(d\mathbf{v}) = \begin{cases} (0,0) & t \leq 0 \\ \chi_{\mathbf{Q}_t}(\mathbf{x}) \cdot (0,1) + \chi_{\mathbf{Q}_{-t}}(\mathbf{x}) \cdot (0,-1) & t > 0. \end{cases} \tag{3.7}$$

The macroscopic density (3.4) and velocity (3.7) show clearly a macroscopic velocity generation (see Figure 1): before $t = 0$, the macroscopic system stays at rest, while, starting at $t = 0$, two equal mass fronts split and move away from each other with velocity ± 1. The sudden increase of macroscopic kinetic energy, of course, comes from interaction with an invisible molecule as we will see in the proof of Theorem 3.1 (subsections 3.1, 3.2, and 3.3). In subsection 3.4 we examine the macroscopic hydrodynamic equation solved by the density (3.4) and velocity (3.7).

3.1. **Interaction with one particle at rest.** Start with two identical molecules P, Q interacting with potential Φ_σ as in (2.1). Denote the positions and velocities of P, Q as $\mathbf{x}_P = (x_P, y_P)$, $\mathbf{x}_Q = (x_Q, y_Q)$, $\mathbf{v}_P$, and $\mathbf{v}_Q$. Consulting Figure 2, let D be the disc with center (x_0, y_0) and radius $r > 0$ and assume that at $t = 0$

(1) $(x_P, y_P) \in D$ and $x_Q = x_0 + d$ with $d > r + \sigma$, i.e. P is inside D and Q is on the vertical line $x = x_0 + d$.
(2) $\mathbf{v}_P = v(\cos\phi, \sin\phi)$ with $-\dfrac{\pi}{2} < \phi < \dfrac{\pi}{2}$, $v > 0$ and $\mathbf{v}_Q = (0,0)$, i.e. P moves with speed v and Q is at rest.

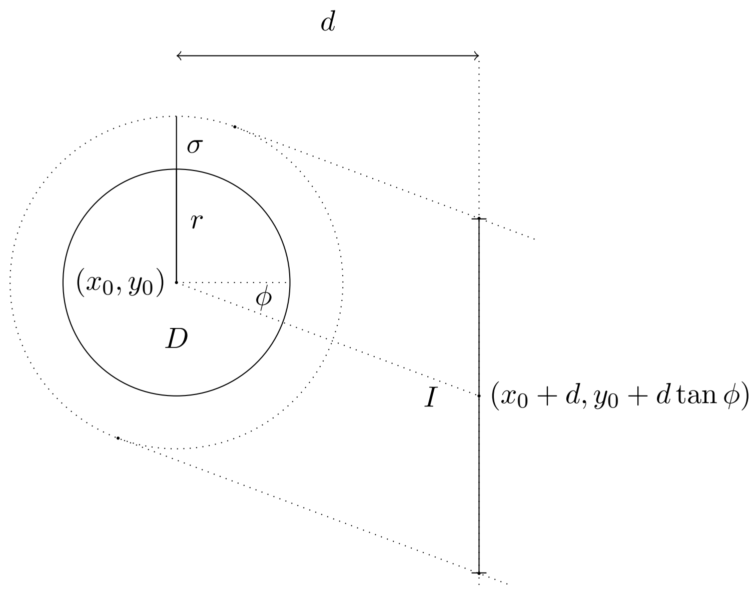

FIGURE 2. The initial disc D and the segment I for $\phi < 0$.

We say that there is interaction between P and Q whenever their distance is smaller than σ. Since Q is at rest at $t = 0$, there will be no interaction between P and Q as long as P is inside D. The following lemma on the interaction between P and Q is the building block of the rest of this section.

Lemma 3.2. *Let P, Q be as above:*

(1) *For any θ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ there exists y_Q such that P and Q will eventually interact (i.e. P and Q will interact at some time $t > 0$), and after interaction P and Q will move in directions perpendicular to each other with constant velocities $\mathbf{v}'_P = v\cos\theta\,(\cos\phi', \sin\phi')$ and $\mathbf{v}'_Q = v\sin\theta\,(\sin\phi', -\cos\phi')$, respectively, where $\phi' = \phi + \theta$.*

(2) *If interaction takes place then y_Q satisfies*

$$|\, y_Q - (y_0 + d\tan\phi)\,| < \frac{r+\sigma}{\cos\phi}.$$

(3) *Whenever P and Q interact, they are both inside the disc with center $(x_0 + d, y_0 + d\tan\phi)$ and radius $\dfrac{r+\sigma}{\cos\phi} + 5\sigma$.*

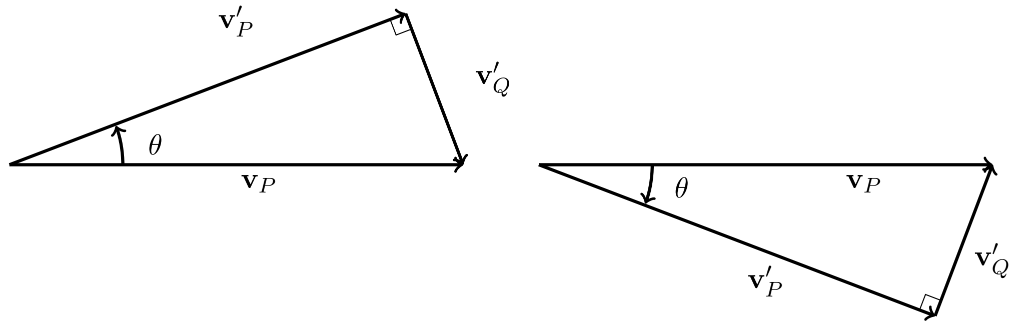

FIGURE 3. The two possible deflection triangles for given $|\mathbf{v}'_Q|$.

Proof. (1) Consulting Figure 3 (which is [LL]'s Figure 17, p. 47, in our notation), for θ the deflection angle from $\mathbf{v}_P$ to $\mathbf{v}'_P$, conservation of momentum and energy gives the formulas of $\mathbf{v}'_P$ and $\mathbf{v}'_Q$. That any θ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is attained by some y_Q follows from Corollary A.3 in the Appendix and the formulas in [LL], §13 that show how to transform from motion in a central field to a system of two molecules.

(2) Let S be the strip between the two lines tangent to D and parallel to $\mathbf{v}_P$, S_σ the set of all points with distance smaller than σ from S, and I be the interval of intersection of S_σ with the line $x = x_0 + d$, see Figure 2. Then if Q has second coordinate anywhere out of I, P ignores it and continues with unaltered velocity $\mathbf{v}_P$. Elementary geometry shows that I has midpoint $y_0 + d\tan\phi$ and half-length $\dfrac{r+\sigma}{\cos\phi}$, consult Figure 2.

(3) By Lemma A.1 in the Appendix, when P and Q interact, their interaction time is less than $\dfrac{4\sigma}{v}$ and by conservation of energy (Φ is positive) the speed of Q will never be more than v during interaction. Therefore, during interaction Q travels less than 4σ, i.e. it stays in the disc centered at $(x_0 + d, y_0 + d\tan\phi)$ with radius $\dfrac{r+\sigma}{\cos\phi} + 4\sigma$. As the distance between P and Q is always less than σ during interaction, P is always inside the circle centered at $(x_0 + d, y_0 + d\tan\phi)$ with radius $\dfrac{r+\sigma}{\cos\phi} + 5\sigma$. $\square$

3.2. **A system of molecules on the plane.** We describe now a system consisting of $N+1$ molecules P, Q_k, $k = 1, \ldots, N$ where P interacts (only once) with each Q_k

(in the order of increasing k) and interactions are independent (P does not interact with P_j, $j \neq k$, when interacting with P_k, and there is no interaction between the Q_k's). In addition, the moment before interacting with Q_k the speed of P will be greater than 1 and the speed of Q_k after interaction will be 1.

We use θ_k for the deflection angle of P due to the interaction with Q_k. Assume that before interacting with Q_1, P moves along the x-axis. Then $\phi_k = \sum_{j=1}^{k} \theta_j$ will be the angle from the x-axis to the direction of the velocity of P right after its interaction with Q_k. The angle from the x-axis to the direction of the velocity of Q_k after its interaction with P will be denoted by $\hat{\phi}_k$. By Figure 3, $\hat{\phi}_k = (-1)^{k+1}\frac{\pi}{2} + \phi_k$.

Lemma 3.3. *For $N \in \mathbb{N}$ fixed and $k = 1, 2, \ldots, N$, let*

$$\theta_k = (-1)^k \arcsin \frac{1}{\sqrt{N+2-k}}, \quad \phi_k = \sum_{j=1}^{k} \theta_j. \tag{3.8}$$

Then

(1) $\phi_k < 0$, *when k is odd and $\phi_k > 0$, when k is even,*
(2) $|\phi_k| < |\phi_{k+2}|$,
(3) $|\phi_1| = |\theta_1| \leq \frac{\pi}{4}$ *and* $|\phi_k| < |\theta_k| \leq \frac{\pi}{4}$, *for $k > 1$.*

Proof of Lemma 3.3. For (1), observe that the θ_k's start negative, increase strictly in absolute value and alternate sign. Therefore for k odd and $k > 1$

$$\phi_k = \theta_1 + (\theta_2 + \theta_3) + \ldots + (\theta_{k-1} + \theta_k) < \theta_1 < 0, \tag{3.9}$$

whereas for k even

$$\phi_k = (\theta_1 + \theta_2) + \ldots + (\theta_{k-1} + \theta_k) > 0. \tag{3.10}$$

For (2), notice that $\theta_{k+1} + \theta_{k+2}$ always has the same sign as ϕ_k, hence

$$|\phi_{k+2}| = |\phi_k| + |\theta_{k+1} + \theta_{k+2}| > |\phi_k|\,. \tag{3.11}$$

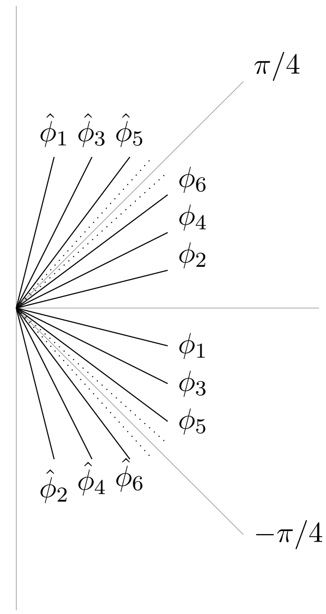

FIGURE 4. The angles ϕ_k and $\hat{\phi}_k$. Observe how the even/odd ϕ_k's and the even/odd $\hat{\phi}_k$'s fall into four non-overlapping sectors.

(3) For $1 \leq k \leq N$

$$|\theta_k| = \arcsin \frac{1}{\sqrt{N+2-k}} \leq \arcsin \frac{1}{\sqrt{2}} = \frac{\pi}{4}. \tag{3.12}$$

For k is odd and $k > 1$

$$|\phi_k| = -\phi_k = -\phi_{k-1} - \theta_k < -\theta_k \leq \frac{\pi}{4}, \tag{3.13}$$

whereas for k even

$$|\phi_k| = \phi_k = \phi_{k-1} + \theta_k < \theta_k \leq \frac{\pi}{4}. \tag{3.14}$$

□

Lemma 3.3 shows that the even ϕ_k's are positive, increasing, and never more than $\pi/4$ (and therefore the even $\hat{\phi}_k$'s are negative, increasing, and never more than $-\pi/4$), whereas the odd ϕ_k's are negative, decreasing, and never less than $-\pi/4$ (and therefore the odd $\hat{\phi}_k$'s are positive, decreasing, and never less than $\pi/4$). Figure 4 summarizes the behavior of ϕ_k and $\hat{\phi}_k$.

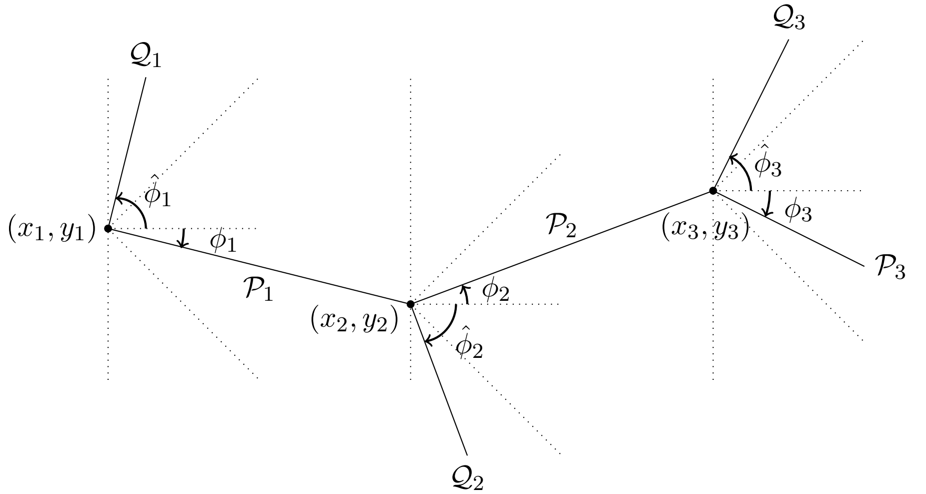

FIGURE 5. First few segments $\mathcal{P}_n$ and half-lines $\mathcal{Q}_n$.

In the description of the interaction of P and Q_k, for ϕ_j as in (3.8) (assuming $\phi_0 = 0$), the point

$$(x_k, y_k) = \left(\frac{k}{N}, \ \frac{1}{N} \sum_{j=0}^{k-1} \tan \phi_j \right) \tag{3.15}$$

will play the same role as $(x_0 + d, y_0 + d \tan \phi)$ in Lemma 3.2. The segments and half-lines

(3.16)

$$\mathcal{P}_k := \{(x, y) : x_k \le x \le x_{k+1}, \ y = (x - x_k) \tan \phi_k + y_k\}, \ k = 1, \dots, N-1,$$

$$\mathcal{P}_N := \{(x, y) : x_N \le x, \ y = (x - x_N) \tan \phi_N + y_N\},$$

$$\mathcal{Q}_k := \left\{(x, y) : x \ge x_k, \ y = (x - x_k) \tan \hat{\phi}_k + y_k\right\}, \ k = 1, \dots, N,$$

will be useful in describing the trajectories of P and of each Q_k, respectively, see Figure 5. Define the distance between any two of these sets as

$$d(\mathcal{A}, \mathcal{B}) := \inf \{\|a - b\| : a \in \mathcal{A}, \ b \in \mathcal{B}\}. \tag{3.17}$$

Lemma 3.4. *Let $m, n = 1, 2, \ldots, N$. For $\mathcal{Q}_m$, $\mathcal{P}_n$ as above,*

$$d\left(\mathcal{Q}_m, \mathcal{Q}_n\right) > \frac{1}{N}, \quad m \neq n, \tag{3.18}$$
$$d\left(\mathcal{Q}_m, \mathcal{P}_n\right) > \frac{1}{N}, \quad m < n.$$

Proof of Lemma 3.4. Recalling (3.16), we use here "right half plane of $\mathcal{Q}_n$" to mean the half-plane to the right of the y-axis defined by: $\mathcal{Q}_n$ is the positive y-axis when $\hat{\phi}_n$ is positive; $\mathcal{Q}_n$ is the negative y-axis when $\hat{\phi}_n$ is negative.

Observe first that for any fixed n, the point (x_n, y_n) is always in the right half plane of $\mathcal{Q}_m$ for all $m < n:$ this holds by the relation of the angles ϕ_i to the angles $\hat{\phi}_j$, see Figure 4 and Figure 5.

To get the first estimate in (3.18), it suffices to consider $n > m$. If $n - m$ is even, then the angle of $\mathcal{Q}_n$ (i.e. $\hat{\phi}_n$) is of smaller absolute value than the angle of $\mathcal{Q}_m$ (i.e. $\hat{\phi}_m$). If $n - m$ is odd, then the angles of $\mathcal{Q}_m$ and $\mathcal{Q}_n$ differ by more than $\pi/2$. In either case the point on $\mathcal{Q}_n$ closest to $\mathcal{Q}_m$ is (x_n, y_n).

Similarly, the angle of $\mathcal{P}_n$ (i.e. ϕ_n), is always of absolute value smaller than the angle of any $\mathcal{Q}_m$ (i.e. $\hat{\phi}_m$). Therefore the point on $\mathcal{P}_n$ closest to $\mathcal{Q}_m$, for $m < n$, is (x_n, y_n).

Now it suffices to notice that the distance from (x_n, y_n) to each Q_m is greater or equal to $|\mathcal{P}_m|$ which is clearly bigger than $\frac{1}{N}$ (consult Figure 4 and Figure 5). $\square$

We are now ready to establish the evolution of $P, Q_1, \ldots, Q_N$.

Proposition 3.5. *For each $N \in \mathbb{N}$ and $\sigma_N < \dfrac{1}{2\sqrt{2}} \dfrac{1}{N(N+3)^{3/2}}$, consider the system $P, Q_1, \ldots, Q_N$ with interaction Φ_{σ_N}. Then there exist y_{Q_k}'s, $k = 1, \ldots, N$, such that the system evolves as follows: for $t \leq 0$,*

$$P(t) = t\mathbf{v}_P, \quad \mathbf{v}_P(t) = \left(\sqrt{N+1}, 0\right), \tag{3.19}$$
$$Q_k(t) = \left(\frac{k}{N}, y_{Q_k}\right), \quad \mathbf{v}_{Q_k}(t) = (0,0), \quad k = 1, \ldots, N,$$

and for $t > 0$,

(1) *There exist times* $0 < t'_1 < t''_1 < t'_2 < t''_2 < \ldots < t'_N < t''_N$ *such that for any* $1 \le k \le N$, P *starts to interact with* Q_k *at* $t = t'_k$ *and completes this interaction at* $t = t''_k$.

(2) *For any* $1 \le k \le N$, *the molecule* Q_k *does not interact with any other molecule for* $t < t'_k$ *or* $t > t''_k$ *and its velocity is given by*

$$\mathbf{v}_{Q_k}(t) = \begin{cases} (0,0) & t < t'_k \\ \big(\sin|\phi_k|, (-1)^{k+1}\cos\phi_k\big) & t > t''_k, \end{cases} \tag{3.20}$$

for ϕ_k *as in* (3.8).

(3) *The velocity of* P *satisfies*

$$\mathbf{v}_P(t) = \begin{cases} (\sqrt{N+1}, 0) & t \le t'_1 \\ \sqrt{N+1-k}\,(\cos\phi_k, \sin\phi_k) & t''_k \le t \le t'_{k+1}, k = 1, 2, \ldots, N-1 \\ (\cos\phi_N, \sin\phi_N) & t \ge t''_N, \end{cases} \tag{3.21}$$

for ϕ_k *as in* (3.8).

(4) *During the time interval* $[t'_k, t''_k]$ *for* $1 \le k \le N$ *the molecules* P *and* Q_k *are in the disc of center* (x_k, y_k) *as in* (3.15) *and radius given recursively by*

$$r_k = \frac{r_{k-1} + \sigma_N}{\cos\phi_{k-1}} + 5\sigma_N, \quad r_0 = 0. \tag{3.22}$$

In particular,

$$r_k < 2\sqrt{2}(N+3)^{3/2}\sigma_N. \tag{3.23}$$

Proof of Proposition 3.5. For all $t \le 0$ and any choice of y_{Q_k}, $k = 1, \ldots, N$, take $\mathbf{v}_p(t) = \big(\sqrt{N+1}, 0\big)$, $P(t) = t\mathbf{v}_P$, $Q_k(t) = \left(\dfrac{k}{N}, y_{Q_k}\right)$, and $\mathbf{v}_{Q_k}(t) = (0,0)$. For all $\sigma_N < \dfrac{1}{N}$, it is clear that $P, Q_1, \ldots, Q_N$ solve the Hamiltonian system for $t \le 0$ (as there is no interaction). We now specify y_{Q_k}'s for the evolution when $t > 0$.

Applying Lemma 3.2 for $x_0 = y_0 = 0$, $r = 0$, $\phi = 0$, $v = \sqrt{N+1}$, $d = 1/N$ and $\theta = \theta_1 = \phi_1 = -\arcsin(1/\sqrt{N+1})$ there is y_{Q_1} such that P will interact with Q_1

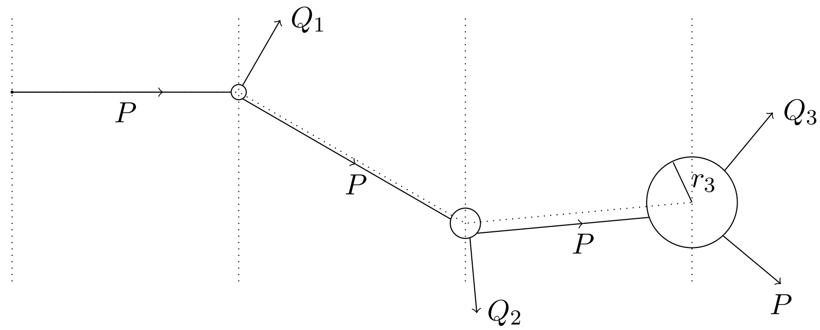

FIGURE 6. Schematics of the system of Proposition 3.5 after three interactions. The radii discs are not up to scale.

and after interaction

$$\mathbf{v}_P = \sqrt{N}\left(\cos\phi_1, \sin\phi_1\right), \quad \mathbf{v}_{Q_1} = \left(-\sin\phi_1, \cos\phi_1\right). \tag{3.24}$$

In this way, the position of Q_1, depending on σ_N, is now determined. The whole interaction, according to Lemma 3.2, takes place in the disc of radius $r_1 = 6\sigma_N$ and center $(1/N, 0)$. Let $[t_1', t_1'']$ be the time interval of this interaction. Preparing for the next interaction, make a new choice of σ_N so that $r_1 = 6\sigma_N < 1/N$, and note that everything in this first step still holds for the new choice of σ_N.

For induction, fix $k \in \mathbb{N}$ and assume that $r_1, \ldots, r_k$ satisfy (3.23), and therefore $r_j < 1/N$, $j = 1, \ldots, k$, for all σ_N small enough. Further assume that $y_{Q_1}, \ldots, y_{Q_k}$, $t_1', \ldots, t_k'$, $t_1'', \ldots, t_k''$, $\mathbf{v}_{Q_1}(t), \ldots, \mathbf{v}_{Q_k}(t)$, $\mathbf{v}_P(t)$, for $t \le t_k''$, have all been determined and satisfy (3.21) and (3.20).

Apply Lemma 3.2 for $(x_0, y_0) = (x_k, y_k)$, for (x_k, y_k) as in (3.15), $r = r_k$, $\phi = \phi_k$, $v = \sqrt{N+1-k}$, $d = 1/N$ and $\theta = \theta_{k+1}$ as in (3.8), to find that r_{k+1} is determined by formula (3.23), to determine $y_{Q_{k+1}}$, the times t_{k+1}', t_{k+1}'', and the velocities $\mathbf{v}_P(t)$, $\mathbf{v}_{Q_{k+1}}(t)$ for $t \in [t_{k+1}', t_{k+1}'']$ that will satisfy (3.21) and (3.20). Therefore Q_{k+1} is always in the r_{k+1}-neighborhood of $\mathcal{Q}_{k+1}$, as defined in (3.16). Choose σ_N so that r_{k+1} is smaller than $1/N$. Using Lemma 3.4, Q_{k+1} does not interact with $Q_1, \ldots, Q_k$ during the interval $(-\infty, t_{k+1}'']$.

For (3.23), rewrite first (3.22) as

$$\begin{aligned} r_k &= \sec\phi_{k-1} r_{k-1} + \sec\phi_{k-1}\sigma_N + 5\sigma_N \\ &= \prod_{j=0}^{k-1} \sec\phi_j r_0 + \sum_{j=0}^{k-1}\prod_{m=j}^{k-1} \sec\phi_m \sigma_N + \left(\sum_{j=1}^{k-1}\prod_{m=j}^{k-1}\sec\phi_m + 1\right) 5\sigma_N \end{aligned} \tag{3.25}$$

and, using $r_0 = 0$ and $|\phi_j| \le |\theta_j|$ (Lemma 3.3), estimate this by

$$\le \sum_{j=0}^{k-1}\prod_{m=j}^{k-1} \sec\theta_m \sigma_N + \left(\sum_{j=1}^{k-1}\prod_{m=j}^{k-1}\sec\theta_m + 1\right) 5\sigma_N, \tag{3.26}$$

and then, increasing k to N and using (3.8), estimate the same by

$$\begin{aligned} &\le \sum_{j=0}^{N-1} \frac{\sqrt{N+2-j}}{\sqrt{2}}\sigma_N + \left(\sum_{j=1}^{N-1}\frac{\sqrt{N+2-j}}{\sqrt{2}} + 1\right) 5\sigma_N \\ &\le 6\sum_{j=0}^{N-1}\frac{\sqrt{N+2-j}}{\sqrt{2}}\sigma_N \le 3\sqrt{2}\sum_{j=3}^{N+2}\sqrt{j}\sigma_N \\ &\le 3\sqrt{2}\sigma_N \int_3^{N+3}\sqrt{x}dx < 2\sqrt{2}(N+3)^{3/2}\sigma_N. \end{aligned} \tag{3.27}$$

In particular, $\sigma_N < \dfrac{1}{2\sqrt{2}}\dfrac{1}{N(N+3)^{3/2}}$ implies $r_k < 1/N$ for all k. □

Remark 3.6. *Notice that, for each N, Proposition 3.5 provides examples of the general theory of Gal'perin and Vaserstein,* [G] *and* [V]*, according to which, for finite range interactions, molecules evolve by eventually separating into independent clusters. Each cluster here consists of a single molecule.*

3.3. **The limit system as $N \to \infty$.** In the notation of Proposition 3.5, let $T'_N := \sum_{j=1}^{N}(t''_j - t'_j)$, the time during which P interacts with some Q_k. Then $T''_N = t''_N - T'_N$ is the time during $[0, t''_N]$ when P is not interacting at all.

Proposition 3.7. $t''_N \to 0$*, as* $N \to \infty$.

Proof. According to (3.21), the speed of P at t'_k is $\sqrt{N+2-k}$. Then, by Lemma A.1,

$$T'_N < \sum_{k=1}^{N} \frac{4\sigma_N}{\sqrt{N+2-k}} = 4\sigma_N \sum_{k=2}^{N+1} \frac{1}{\sqrt{k}}. \tag{3.28}$$

After the interaction of P with Q_k is complete, P moves with speed $\sqrt{N+1-k}$, forming angle ϕ_k with the x-axis. The distance d_k that P will travel until its interaction with Q_{k+1} begins, satisfies

$$d_k \leq \frac{1}{N\cos\phi_k} \leq \frac{1}{N\cos\theta_k}, \tag{3.29}$$

cf. Figure 2. Recalling that $|\theta_k| \leq \frac{\pi}{4}$ from Lemma 3.3 gives

$$\begin{aligned} T''_N &< \sum_{k=0}^{N-1} \frac{1}{N\cos\theta_k} \frac{1}{\sqrt{N+1-k}} \\ &\leq \frac{\sqrt{2}}{N} \sum_{k=0}^{N-1} \frac{1}{\sqrt{N+1-k}} = \frac{\sqrt{2}}{N} \sum_{k=2}^{N+1} \frac{1}{\sqrt{k}}. \end{aligned} \tag{3.30}$$

This and (3.28) imply

$$t''_N < \left(4\sigma_N + \frac{\sqrt{2}}{N}\right) \sum_{k=2}^{N+1} \frac{1}{\sqrt{k}}. \tag{3.31}$$

As $\sum_{k=2}^{N+1} \frac{1}{\sqrt{k}} < 2\sqrt{N+1}$, and for σ_N as in Proposition 3.5, we conclude that $t''_N \to 0$ as $N \to \infty$. □

Proposition 3.8. $\max_{0\leq k\leq N} y_{Q_k} \to 0$ *as* $N \to \infty$*.*

Proof. Noting that y_{Q_k} is the second coordinate of Q_k before $t = t'_k$, whereas y_k is the second coordinate of the center of the k-interaction disc, it follows from the definition of r_k and (3.23) that

$$|y_{Q_k}| < |y_k| + 2\sqrt{2}(N+3)^{3/2}\sigma_N. \tag{3.32}$$

For the second term on the right use σ_N as in Proposition 3.5 and estimate the first term as

$$
\begin{aligned}
|y_k| &= \frac{1}{N}\left|\sum_{m=0}^{k-1}\tan\phi_m\right| \le \frac{1}{N}\sum_{m=0}^{k-1}\tan|\phi_m| \\
&\le \frac{1}{N}\sum_{m=0}^{k-1}\tan|\theta_m| = \frac{1}{N}\sum_{m=0}^{k-1}\tan\left(\arcsin\frac{1}{\sqrt{N+2-m}}\right) \\
&= \sum_{m=0}^{k-1}\frac{1}{N\sqrt{N+1-m}} < \sum_{m=0}^{N-1}\frac{1}{N\sqrt{N+1-m}} \\
&= \frac{1}{N}\sum_{m=2}^{N+1}\frac{1}{\sqrt{m}} \to 0,
\end{aligned} \tag{3.33}
$$

as $N \to \infty$. □

For each fixed N, writing $\mathbf{v} = (v_x, v_y)$ and following (2.3), set for $t \in \mathbb{R}$

$$
\begin{aligned}
&M_t^{(N+1)}(dx, dy, dv_x, dv_y) \\
&\quad = \frac{1}{N+1}\left(\delta_{(P(t),\mathbf{v}_P(t))}(dx, dy, dv_x, dv_y) + \sum_{k=1}^{N}\delta_{\left(Q_k(t),\mathbf{v}_{Q_k}(t)\right)}(dx, dy, dv_x, dv_y)\right).
\end{aligned} \tag{3.34}
$$

The crucial observation in the following proposition is that, due to the factor $1/N$, no single molecule shows as $N \to \infty$, but its interaction with many other molecules, if their number is of order N, shows macroscopically.

Proposition 3.9. *As $N \to \infty$, and for σ_N as in Proposition 3.5: for $t \le 0$,*

$$M_t^{(N+1)}(dx, dy, dv_x, dv_y) \Rightarrow \chi_{[0,1]}(x)dx \otimes \delta_0(dy) \otimes \delta_{(0,0)}(dv_x, dv_y), \tag{3.35}$$

and for $t > 0$,

$$
\begin{aligned}
&M_t^{(N+1)}(dx, dy, dv_x, dv_y) \\
&\quad \Rightarrow \chi_{[0,1]}(x)dx \otimes \left(\frac{1}{2}\delta_t(dy) \otimes \delta_{(0,1)}(dv_x, dv_y) + \frac{1}{2}\delta_{-t}(dy) \otimes \delta_{(0,-1)}(dv_x, dv_y)\right).
\end{aligned} \tag{3.36}
$$

Proof. It suffices to check the statement on the integrals of bounded Lipschitz functions, see [AGS], page 109. For this, for $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ bounded and Lipschitz

$$\begin{aligned} &\int_{\mathbb{R}^4} f\left(x, y, v_x, v_y\right) M_t^{(N+1)}(dx, dy, dv_x, dv_y) \\ &\qquad = \frac{1}{N+1} f\left(P(t), \mathbf{v}_P(t)\right) + \frac{1}{N+1} \sum_{k=1}^{N} f\left(Q_k(t), \mathbf{v}_{Q_k}(t)\right). \end{aligned} \tag{3.37}$$

Since f is bounded, the first term vanishes as $N \to \infty$. The rest of the proof examines the convergence of the second term.

Fix any $t \leq 0$. Recalling (3.19),

$$\frac{1}{N+1} \sum_{k=1}^{N} f\left(Q_k(t), \mathbf{v}_{Q_k}(t)\right) = \frac{1}{N+1} \sum_{k=1}^{N} f\left(\frac{k}{N}, y_{Q_k}, 0, 0\right). \tag{3.38}$$

For L_f be the Lipschitz constant of f, and using Proposition 3.8,

$$\begin{aligned} &\left| \frac{1}{N+1} \sum_{k=1}^{N} f\left(\frac{k}{N}, y_{Q_k}, 0, 0\right) - \frac{1}{N+1} \sum_{k=1}^{N} f\left(\frac{k}{N}, 0, 0, 0\right) \right| \\ &\qquad \leq \frac{N}{N+1} L_f \max_{1 \leq k \leq N} |y_{Q_k}| \to 0. \end{aligned} \tag{3.39}$$

By the definition of the Riemann integral,

$$\frac{1}{N+1} \sum_{k=1}^{N} f\left(\frac{k}{N}, 0, 0, 0\right) \to \int_0^1 f\left(x, 0, 0, 0\right) dx. \tag{3.40}$$

Therefore

$$\int_{\mathbb{R}^4} f\left(x, y, v_x, v_y\right) M_t^{(N+1)}(dx, dy, dv_x, dv_y) \to \int_0^1 f\left(x, 0, 0, 0\right) dx. \tag{3.41}$$

This is exactly (3.35). Now fix $t > 0$. By Proposition 3.7 there exists N_1 such that for all $N > N_1$, $t_N'' < t$, i.e. for each time we can choose N large enough so that all interactions have already happened and all molecules *are* moving at time t, and are moving with their terminal velocities. We consider such N's only. According

to Proposition 3.5, and since now $t \geq t_k''$,

$$
\begin{aligned}
x_{Q_k}(t) &= x_{Q_k}(t_k'') + \left(t - t_k''\right) v_{Q_k,x}(t_k''), \\
y_{Q_k}(t) &= y_{Q_k}(t_k'') + \left(t - t_k''\right) v_{Q_k,y}(t_k'').
\end{aligned} \tag{3.42}
$$

For $\alpha_N = \left[N - \sqrt{N}\right]$, the integer part of $N - \sqrt{N}$, and by (3.20), for any $1 \leq k \leq \alpha_N$

$$
\begin{aligned}
&\left|v_{Q_k,x}(t_k'')\right| = \sin|\phi_k| \leq \sin|\theta_k| \leq \frac{1}{\sqrt{N+2-\alpha_N}}, \\
&\left|v_{Q_k,y}(t_k'') - (-1)^{k+1}\right| = |\cos\phi_k - 1| \leq |\sin\phi_k| \leq \frac{1}{\sqrt{N+2-\alpha_N}}, \\
&\left|x_{Q_k}(t_k'') - \frac{k}{N}\right| \leq r_k, \quad \left|y_{Q_k}(t_k'')\right| < |y_k| + r_k.
\end{aligned} \tag{3.43}
$$

Therefore for $1 \leq k \leq \alpha_N$, by (3.23), Proposition 3.7, and Proposition 3.8,

$$
\begin{aligned}
\left|x_{Q_k}(t) - \frac{k}{N}\right| &\leq \left|x_{Q_k}(t_k'') - \frac{k}{N}\right| + \left(t - t_k''\right)\left|v_{x,Q_k}(t_k'')\right| \\
&< r_k + \frac{t}{\sqrt{N+2-\alpha_N}} \to 0, \\
\left|y_{Q_k}(t) - (-1)^{k+1}t\right| &\leq \left|y_{Q_k}(t_k'')\right| + t\left|v_{Q_k,y}(t_k'') - (-1)^{k+1}\right| + t_k''\left|v_{Q_k,y}(t_k'')\right| \\
&< |y_k| + r_k + \frac{t}{\sqrt{N+2-\alpha_N}} + t_k'' \to 0.
\end{aligned} \tag{3.44}
$$

Since f is Lipschitz, (3.43) and (3.44) imply that

$$
\left|\frac{1}{N+1}\sum_{k=1}^{\alpha_N} f\left(Q_k(t), \mathbf{v}_{Q_k}(t)\right) - \frac{1}{N+1}\sum_{k=1}^{\alpha_N} f\left(\frac{k}{N}, (-1)^{k+1}t, 0, (-1)^{k+1}\right)\right| \to 0. \tag{3.45}
$$

For $C_f = \max|f|$,

$$
\begin{aligned}
&\left|\frac{1}{N+1}\sum_{k=\alpha_N+1}^{N} f\left(Q_k(t), \mathbf{v}_{Q_k}(t)\right)\right| \leq C_f\frac{N-\alpha_N}{N+1} \to 0, \\
&\left|\frac{1}{N+1}\sum_{k=\alpha_N+1}^{N} f\left(\frac{k}{N}, (-1)^{k+1}t, 0, (-1)^{k+1}\right)\right| \leq C_f\frac{N-\alpha_N}{N+1} \to 0,
\end{aligned} \tag{3.46}
$$

therefore,

(3.47)
$$\left| \frac{1}{N+1} \sum_{k=1}^{N} f\left(Q_k(t), \mathbf{v}_{Q_k}(t)\right) - \frac{1}{N+1} \sum_{k=1}^{N} f\left(\frac{k}{N}, (-1)^{k+1}t, 0, (-1)^{k+1}\right) \right| \to 0.$$

By the definition of the Riemann integral,

(3.48)
$$\frac{1}{N+1} \sum_{k=1}^{N} f\left(\frac{k}{N}, (-1)^{k+1}t, 0, (-1)^{k+1}\right) \to \int_0^1 \frac{1}{2}\left(f(x,t,0,1) + f(x,-t,0,-1)\right) dx$$

which implies (3.36). □

With

$$\begin{aligned} &\mathbf{x}_k^{(N+1)}(t) = Q_k(t), \quad \mathbf{u}_k^{(N+1)}(t) = \mathbf{v}_{Q_k}(t), \ k = 1, \dots, N \\ &\mathbf{x}_{N+1}^{(N+1)}(t) = P(t), \quad \mathbf{u}_{N+1}^{(N+1)}(t) = \mathbf{v}_P(t), \end{aligned} \tag{3.49}$$

Theorem 3.1 follows immediately from Propositions 3.5 and 3.9.

3.4. **Macroscopic equations.** We now examine the hydrodynamic equations for $M_t(d\mathbf{x}, d\mathbf{v})$ as in Theorem 3.1. It is easy to check that for any $\phi(t, \mathbf{x}) \in C_c^\infty(\mathbb{R} \times \mathbb{R}^2)$

(3.50)
$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{\mathbb{R}^4} \partial_t \phi(t, \mathbf{x}) M_t(d\mathbf{x}, d\mathbf{v}) dt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^4} \nabla_{\mathbf{x}} \phi(t, \mathbf{x}) \cdot \mathbf{v} M_t(d\mathbf{x}, d\mathbf{v}) dt = 0, \\ &\int_{-\infty}^{\infty} \int_{\mathbb{R}^4} \partial_t \phi(t, \mathbf{x}) \mathbf{v} M_t(d\mathbf{x}, d\mathbf{v}) dt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^4} \nabla_{\mathbf{x}} \phi(t, \mathbf{x}) \cdot \mathbf{v}\, \mathbf{v} M_t(d\mathbf{x}, d\mathbf{v}) dt = 0. \end{aligned}$$

Using disintegration (3.5), for $\mu_t(d\mathbf{x})$ and $\mathbf{u}(t, \mathbf{x})$ as in (3.4) and (3.7), we rewrite (3.50) as

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \partial_t \phi(t, \mathbf{x}) \mu_t(d\mathbf{x}) dt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \nabla_{\mathbf{x}} \phi(t, \mathbf{x}) \cdot \mathbf{u}(t, \mathbf{x}) \mu_t(d\mathbf{x}) dt = 0, \\ &\int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \partial_t \phi(t, \mathbf{x}) \mathbf{u}(t, \mathbf{x}) \mu_t(d\mathbf{x}) dt \\ &\qquad + \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \nabla_{\mathbf{x}} \phi(t, \mathbf{x}) \cdot \left(\int_{\mathbb{R}^2} \mathbf{v} \otimes \mathbf{v} M_{t,\mathbf{x}}(d\mathbf{v}) \right) \mu_t(d\mathbf{x}) dt = 0. \end{aligned} \tag{3.51}$$

Notice that at each $t, \mathbf{x}$ the $M_{t,\mathbf{x}}(d\mathbf{v})$ is singular, therefore

$$\int_{\mathbb{R}^2} \mathbf{v} \otimes \mathbf{v} M_{t,\mathbf{x}}(d\mathbf{v}) = \mathbf{u} \otimes \mathbf{u}. \tag{3.52}$$

Then (3.51) becomes

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \partial_t \phi(t,\mathbf{x}) \mu_t(d\mathbf{x}) dt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \nabla_{\mathbf{x}} \phi(t,\mathbf{x}) \cdot \mathbf{u}\, \mu_t(d\mathbf{x}) dt = 0, \\ &\int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \partial_t \phi(t,\mathbf{x}) \mathbf{u}\, \mu_t(d\mathbf{x}) dt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \nabla_{\mathbf{x}} \phi(t,\mathbf{x}) \cdot \mathbf{u}\, \mathbf{u}\, \mu_t(d\mathbf{x}) dt = 0. \end{aligned} \tag{3.53}$$

In other words $(\mu_t(d\mathbf{x}), \mathbf{u}(t,\mathbf{x}))$, $t \in \mathbb{R}$ solves weakly two dimensional Euler system without pressure:

$$\begin{aligned} &\partial_t \mu_t + \operatorname{div}(\mathbf{u}\, \mu_t) = 0, \\ &\partial_t(\mathbf{u}\, \mu_t) + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}\ \mu_t) = 0. \end{aligned} \tag{3.54}$$

For the naturalness of measure solutions in the presureless Euler system, see [ERS], p. 354.

Remark 3.10. *The trivial solution* $\widetilde{\mu}_t(d\mathbf{x}) = \Delta_0(d\mathbf{x}), \widetilde{\mathbf{u}} = (0,0)$ *also solves* (3.53) *for all* t*, and coincides with* $(\mu_t, \mathbf{u})$ *for* $t \leq 0$*. Note that* $(\mu_t, \mathbf{u})$ *is not "energy admissible" since the kinetic energy of* $(\mu_t, \mathbf{u})$ *increases in time:*

$$\int_{\mathbb{R}^2} |\mathbf{u}|^2 \mu_t(d\mathbf{x}) = \int_{\mathbb{R}^4} |\mathbf{v}|^2 M_t(d\mathbf{x}, d\mathbf{v}) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0. \end{cases} \tag{3.55}$$

A solution to (3.53) *with decreasing energy can be obtained by reversing the direction of time, as in the next section. The value of the construction in this section lies in the microscopic, Hamiltonian interpretation of spontaneous velocity generation in weak solutions of hydrodynamic equations as in* [Sch], [Sh].

4. Time Reversal and Macroscopic Non-Uniqueness

4.1. **Reverse flow with decreasing energy.** We now reverse time in the construction of the previous section to establish macroscopic non-uniqueness in the class of energy decreasing solutions. It is standard that for $\left(\mathbf{x}_k^{(N)}(t), \mathbf{u}_k^{(N)}(t)\right)$ a Hamiltonian flow as in Theorem 3.1 the reverse flow $\left(\mathbf{x}_k^{(N)}(-t), -\mathbf{u}_k^{(N)}(-t)\right)$ also

$t < 0$ $\qquad\qquad$ $t \geq 0$

FIGURE 7. Macroscopic flow of (4.1).

solves the Hamiltonian system (2.1). Roughly speaking, for each N the reverse system consists of N molecules moving with speed 1 for $t < 0$. At $t = 0$, through interaction, one of the N molecules gathers all the energy from the rest $N-1$ molecules and leaves the rest of the group. Therefore for $t > 0$, macroscopically the system is motionless. If we still use $\left(\mathbf{x}_k^{(N)}(t), \mathbf{u}_k^{(N)}(t)\right)$ for the reverse flow then the measure $M_t^{(N)}$ converges weakly to

$$M_t(d\mathbf{x}, d\mathbf{v}) = \begin{cases} \frac{1}{2}\Delta_t(d\mathbf{x}) \otimes \delta_{(0,1)}\,(d\mathbf{v}) + \frac{1}{2}\Delta_{-t}(d\mathbf{x}) \otimes \delta_{(0,-1)}\,(d\mathbf{v}) & t < 0 \\ \Delta_0(d\mathbf{x}) \otimes \delta_{(0,0)}\,(d\mathbf{v}) & t \geq 0, \end{cases} \tag{4.1}$$

with

$$\begin{aligned} \mu_t(d\mathbf{x}) &= \begin{cases} \frac{1}{2}\Delta_t(d\mathbf{x}) + \frac{1}{2}\Delta_{-t}(d\mathbf{x}) & t < 0 \\ \Delta_0(d\mathbf{x}) & t \geq 0, \end{cases} \\ \mathbf{u}(t, \mathbf{x}) &= \begin{cases} \chi_{\mathbf{Q}_t}(\mathbf{x}) \cdot (0,1) + \chi_{\mathbf{Q}_{-t}}(\mathbf{x}) \cdot (0,-1) & t < 0 \\ 0 & t \geq 0, \end{cases} \end{aligned} \tag{4.2}$$

and decreasing energy:

$$\int_{\mathbb{R}^2} |\mathbf{u}|^2 \mu_t(d\mathbf{x}) = \int_{\mathbb{R}^4} |\mathbf{v}|^2 M_t(d\mathbf{x}, d\mathbf{v}) = \begin{cases} 1 & t < 0 \\ 0 & t \geq 0, \end{cases} \tag{4.3}$$

cf. [BN], Defintion 2.1.

Remark 4.1. *This describes two fronts approaching each other up until* $t = 0$*, when they merge and stay at rest, see Figure 7. In the context of the pressureless Euler system this is a "sticky" macroscopic solution, cf.* [BN]*. Rather than using*

particle systems with adhesion dynamics, here we obtain the solution as the limit of Hamiltonian dynamics with repulsive force. We also provide an explanation for the loss of energy: all the energy is transferred to a macroscopically invisible part of the system.

4.2. **Transverse flow.** It is known that merely requiring decreasing energy does not guarantee uniqueness of measure solutions to the system (3.54), see [BN]. This persists when comparing the flow of the previous section with the limit of a trivial Hamiltonian flow: for this we take the N-system to consist of molecules that stay far enough from each other so that they never interact. We obtain a solution to the system (3.53) that coincides with (4.2) for all $t < 0$. But at $t = 0$, the moment the two fronts meet, instead of merging and staying at rest, they go through each other.

More precisely, for each $N = 2n \in \mathbb{N}$, $j = 1, 2, \ldots, N$, let

$$\widetilde{\mathbf{x}}_j^{(N)} = \left(\frac{j}{N}, 0\right), \quad \widetilde{\mathbf{u}}_j^{(N)} = \begin{cases} (0,1) & \text{if } j \text{ odd} \\ (0,-1) & \text{if } j \text{ even.} \end{cases} \tag{4.4}$$

For $t \in \mathbb{R}$ the orbits

$$\widetilde{\mathbf{x}}_j^{(N)}(t) = \widetilde{\mathbf{x}}_j^{(N)} + t\widetilde{\mathbf{u}}_j^{(N)} \tag{4.5}$$

satisfy the Hamiltonian system (2.1) provided that the interaction range is sufficiently short, for example, $\sigma < 1/N$. (Notice that σ_N in Theorem 3.1, and therefore in Section 4 satisfies $\sigma_N < 1/N$.) Recalling definition (2.3), set

$$\widetilde{M}_t^{(N)}(d\mathbf{x}, d\mathbf{v}) = \frac{1}{N}\sum_{j=1}^{N} \delta_{\left(\widetilde{\mathbf{x}}_j^{(N)}(t), \widetilde{\mathbf{u}}_j^{(N)}(t)\right)}(d\mathbf{x}, d\mathbf{v}). \tag{4.6}$$

By the definition of Riemann integral, for any continuous bounded $f(\mathbf{x}, \mathbf{v})$ we have

(4.7)

$$\lim_{N\to\infty} \int_{\mathbb{R}^4} f(\mathbf{x}, \mathbf{v}) \widetilde{M}_t^{(N)}(d\mathbf{x}, d\mathbf{v}) = \frac{1}{2}\int_0^1 f(x, t, 0, 1)\, dx \; + \frac{1}{2}\int_0^1 f(x, -t, 0, -1)\, dx.$$

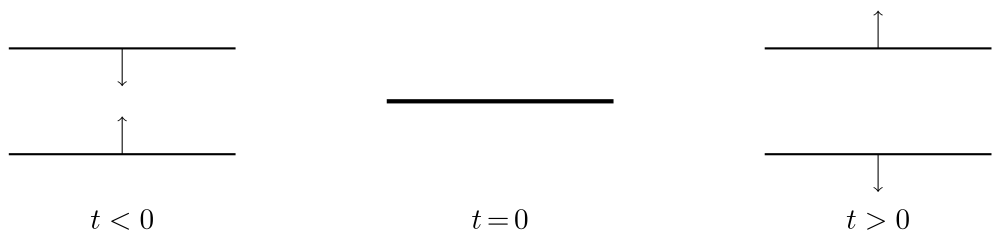

Figure 8. Macroscopic flow of (4.9).

Therefore

$$\begin{aligned}\widetilde{M}_t^{(N)}(d\mathbf{x}, d\mathbf{v}) &\Rightarrow \widetilde{M}_t(d\mathbf{x}, d\mathbf{v}) \\ &:= \frac{1}{2}\Delta_t(d\mathbf{x}) \otimes \delta_{(0,1)}\,(d\mathbf{v}) + \frac{1}{2}\Delta_{-t}(d\mathbf{x}) \otimes \delta_{(0,-1)}\,(d\mathbf{v})\,.\end{aligned} \tag{4.8}$$

The macroscopic density and velocity are

$$\begin{aligned}\widetilde{\mu}_t(d\mathbf{x}) &= \frac{1}{2}\Delta_t(d\mathbf{x}) + \frac{1}{2}\Delta_{-t}(d\mathbf{x}), \\ \widetilde{\mathbf{u}}(t, \mathbf{x}) &= \chi_{\mathbf{Q}_t}(\mathbf{x}) \cdot (0,1) + \chi_{\mathbf{Q}_{-t}}(\mathbf{x}) \cdot (0,-1), \quad t \in \mathbb{R},\end{aligned} \tag{4.9}$$

see Figure 8. It is easily checked that (3.50), (3.51) hold, and that for all $t \neq 0$

$$\int_{\mathbb{R}^2} \mathbf{v} \otimes \mathbf{v} \widetilde{M}_{t,\mathbf{x}}(d\mathbf{v}) = \widetilde{\mathbf{u}} \otimes \widetilde{\mathbf{u}}. \tag{4.10}$$

Therefore $(\widetilde{\mu}_t(d\mathbf{x}), \widetilde{\mathbf{u}}(t, \mathbf{x}))$ also solves weakly the pressureless Euler system for $t \in \mathbb{R}$. Since $\int_{\mathbb{R}^2} |\widetilde{\mathbf{u}}|^2 \widetilde{\mu}_t(d\mathbf{x}) = 1$ except for $t = 0$, we can alter $\widetilde{\mathbf{u}}$ at time $t = 0$ so that

$$\int_{\mathbb{R}^2} |\widetilde{\mathbf{u}}|^2 \widetilde{\mu}_0(d\mathbf{x}) = 1, \tag{4.11}$$

still solving equation (3.53). If we still use $\widetilde{\mu}_t(d\mathbf{x}), \widetilde{\mathbf{u}}(t, \mathbf{x})$ for the modified solution, we then have constant macroscopic kinetic energy in time:

$$\int_{\mathbb{R}^2} |\widetilde{\mathbf{u}}|^2 \widetilde{\mu}_t(d\mathbf{x}) = 1, \quad t \in \mathbb{R}. \tag{4.12}$$

Clearly for all $t < 0$, $(\widetilde{\mu}_t(d\mathbf{x}), \widetilde{\mathbf{u}}(t, \mathbf{x}))$, modified or not, coincides with $(\mu_t(d\mathbf{x}), \mathbf{u}(t, \mathbf{x}))$. Macroscopically, the same two fronts are approaching each other and, unless we know their microscopic origin, we are not be able to tell what will happen for $t > 0$.

Remark 4.2. *Notice here the* total *macroscopic energy of the limit system is conserved in time:*

$$\int_{\mathbb{R}^4} |\mathbf{v}|^2 \widetilde{M_t}(d\mathbf{x}, d\mathbf{v}) = 1, \quad t \in \mathbb{R}, \tag{4.13}$$

and the macroscopic kinetic energy $\int_{\mathbb{R}^2} |\widetilde{\mathbf{u}}|^2 \widetilde{\mu}_t(d\mathbf{x})$ *is only part of the total energy in general:*

$$\int_{\mathbb{R}^4} |\mathbf{v}|^2 \widetilde{M_t}(d\mathbf{x}, d\mathbf{v}) = \int_{\mathbb{R}^2} |\widetilde{\mathbf{u}}|^2 \mu_t(d\mathbf{x}) + \int_{\mathbb{R}^4} |\mathbf{v} - \widetilde{\mathbf{u}}|^2 \widetilde{M_t}(d\mathbf{x}, d\mathbf{v}). \tag{4.14}$$

Let $h(t) = \int_{\mathbb{R}^4} |\mathbf{v} - \widetilde{\mathbf{u}}|^2 \, \widetilde{M_t}(d\mathbf{x}, d\mathbf{v})$. *Then*

$$\int_{\mathbb{R}^2} |\widetilde{\mathbf{u}}|^2 \widetilde{\mu}_t(d\mathbf{x}) + h(t) = 1, \quad t \in \mathbb{R}. \tag{4.15}$$

Notice that $h(t) = 0$ *when* $t \neq 0$ *and* $h(0) = 1$. *Therefore for* $t < 0$, *all the energy of the system* (4.8) *is macroscopic kinetic energy which becomes* $h(0)$, *the fluctuation energy, at* $t = 0$. *For* $t > 0$ *all the energy is again macroscopic kinetic energy.*

By (4.3), *for the reverse flow in Section 4.1, the total energy* $\int_{\mathbb{R}^4} |\mathbf{v}|^2 M_t(d\mathbf{x}, d\mathbf{v})$ *is decreasing in time. Trivially, the corresponding fluctuation energy* $h(t) = 0$ *for all* $t \in \mathbb{R}$.

Remark 4.3. *It is possible that from a Statistical Mechanics point of view the non-uniqueness described here can be avoided by excluding a set of flows* M_t *negligible with respect to some probability measure. Notwithstanding this, our aim here is to understand specific non-uniqueness examples.*

5. Non-Uniqueness from Moments of Measures Satisfying Identical Transport Equations

Section 4 has shown non-uniqueness by comparing moments of the two limit flows $M_t(d\mathbf{x}, d\mathbf{v})$ of (4.1) and $\widetilde{M_t}(d\mathbf{x}, d\mathbf{v})$ of (4.8). Note that M_t satisfies weakly the transport equation

$$\partial_t M_t + \mathbf{v} \cdot \nabla_{\mathbf{x}} M_t = 0, \tag{5.1}$$

while $\widetilde{M}_t$ satisfies the same with a nonzero kick at $t = 0$:

$$\partial_t \widetilde{M}_t + \mathbf{v} \cdot \nabla_{\mathbf{x}} \widetilde{M}_t = \left(\widetilde{M}_{0^+} - \widetilde{M}_{0^-} \right) \otimes \delta_0(dt), \quad t \in \mathbb{R}, \tag{5.2}$$

for $\widetilde{M}_{0^\pm} = \lim_{t \to 0^\pm} \widetilde{M}_t$. In this section we present two examples where two different measures solve the same transport equation (5.1), give identical macroscopic density and velocity at $t = 0$, but the macroscopic density and velocity evolve differently to provide a non-uniqueness result for the Cauchy problem of the compressible Euler system in space dimension one.

5.1. **Finite systems with velocity exchange.** For systems in space dimension 1, we use identical molecules that move freely until they collide. The arguments in this section also hold for systems (2.1) of (finite range, at least) interactions, rescaled as in (2.2). In fact, there exist σ_N's such that, for space dimension 1, the limit of elastic collisions coincides with the limit of rescaled interactions, see [X]. However, such σ_N's might be too small for the rescaled interaction model to be physically better than elastic collisions. For simplicity then, we shall use elastic collisions. The complications of finite range interactions were evident in Section 3.

In the elastic collision model collisions are instantaneous. Momentum and energy are conserved. Here it will be enough to consider only two kinds of collisions, both compatible with finite range interaction dynamics:

(1) Binary collisions with incoming velocities v_1, v_2 and outgoing velocities v_1', v_2' satisfying

$$\left. \begin{aligned} v_1 + v_2 &= v_1' + v_2' \\ v_1^2 + v_2^2 &= (v_1')^2 + (v_2')^2 \end{aligned} \right\} \Rightarrow v_1 = v_2', \ v_2 = v_1', \tag{5.3}$$

i.e. the molecules exchange velocities (as they are not allowed to go through each other).

(2) Triple collisions, consisting of two molecules exactly as in item (1) and a third molecule in between that stays motionless.

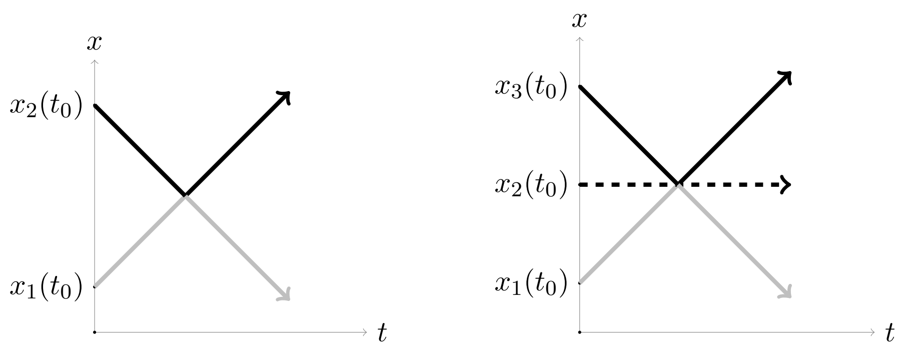

FIGURE 9. The collisions of subsection 5.1.

As Zemlyakov shows in his delightful article [Z], several important questions for such systems can be answered using the graphs of the molecule positions as functions of time. Following this, the two types of collision we consider are shown in Figure 9.

Consider a 1-dimensional point system $\left(x_k^{(N)}(t), u_k^{(N)}(t)\right)$, $k = 1, \ldots, N$ obeying elastic collision dynamics. Fix any $T \in (0, \infty)$. For all $t \in [0, T]$, assume that all collisions are binary or triple as above.

Proposition 5.1. *Let $S_t(x, v) = (x + vt, v)$. For all $t \in [0, T]$ the empirical measures*

$$M_t(dx, dv) = \frac{1}{N} \sum_{k=1}^{N} \delta_{\left(x_k^{(N)}(t), u_k^{(N)}(t)\right)}(dx, dv) \tag{5.4}$$

satisfy

$$M_t^{(N)}(dx, dv) = S_t M_0^{(N)}(dx, dv). \tag{5.5}$$

Proof. Merely notice that for each t

$$\frac{1}{N} \sum_{k=1}^{N} \delta_{\left(x_k^{(N)}(t), u_k^{(N)}(t)\right)} = \frac{1}{N} \sum_{k=1}^{N} \delta_{\left(x_k^{(N)}(0) + t u_k^{(N)}(0), u_k^{(N)}(0)\right)} \tag{5.6}$$

since there is a bijection, if multiplicities are taken into account:

$$\left\{\left(x_k^{(N)}(t), u_k^{(N)}(t)\right)\right\} \leftrightarrow \left\{\left(x_k^{(N)}(0) + t u_k^{(N)}(0), u_k^{(N)}(0)\right)\right\}. \tag{5.7}$$

Indeed, the exchange of velocities between the moving molecules of a collision establishes a bijection between the orbits before and after that collision. Iterating this finitely many times brings us back to the initial orbits given by $\left(x_k^{(N)}(0) + tu_k^{(N)}(0), u_k^{(N)}(0)\right)$. □

The following Lemma will be used repeatedly.

Lemma 5.2. *Suppose that*

$$M_t^{(N)}(dx, dv) = S_t M_0^N(dx, dv), \quad M_0^{(N)}(dx, dv) \Rightarrow M_0(dx, dv). \tag{5.8}$$

Then $M_t^{(N)}(dx, dv) \Rightarrow S_t M_0(dx, dv)$.

Proof. Use the definitions of weak convergence and push forward under S_t. □

As it is standard that $M_t(dx, dv) = S_t M_0(dx, dv)$ solves weakly the free transport equation

$$\partial_t M_t + v\partial_x M_t = 0 \tag{5.9}$$

we shell refer to it as the *a free transport flow.*

5.2. **Euler system from free transport flow.** We find here conditions that imply that averages with respect to free transport flow satisfy the compressible Euler system in dimension 1. The next two subsections provide examples satisfying such conditions.

Lemma 5.3. *Suppose that* $M_t(dx, dv) = S_t M_0(dx, dv)$. *Then for all* $\phi(t, x) \in C_c^1([0, T) \times \mathbb{R})$ *and* $g(v)$ *such that* $vg(v) \in L^1(M_0)$*, we have*

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^2} [\partial_t \phi(t, x) g(v) + \partial_x \phi(t, x)\, vg(v)] M_t(dx, dv) dt \\ + \int_{\mathbb{R}^2} \phi(0, x) g(v) M_0(dx, dv) = 0.\end{aligned} \tag{5.10}$$

Proof. Straight forward calculation using the definition of the push forward under S_t and the assumption that ϕ is compactly supported. □

Disintegrating $M_t(dx, dv)$ of Lemma 5.3 as

$$M_t(dx, dv) = \int M_{t,x}(dv)\mu_t(dx), \tag{5.11}$$

and for

$$\overline{g(v)}(t, x) = \int g(v) M_{t,x}(dv), \tag{5.12}$$

(5.10) becomes

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} \left[\partial_t\phi(t, x)\, \overline{g(v)}(t, x) + \partial_x\phi(t, x)\, \overline{vg(v)}(t, x)\right] \mu_t(dx)dt \\ + \int_{\mathbb{R}} \phi(0, x)\, \overline{g(v)}(0, x)\mu_0(dx) = 0.\end{aligned} \tag{5.13}$$

To apply Lemma 5.3 for $g(v) = 1, v$, and $\frac{1}{2}v^2$, assume $v^3 \in L^1(M_0)$. Noting that

$$u(t, x) = \overline{v}(t, x) = \int v M_{t,x}(dv), \tag{5.14}$$

and using the notation

$$\overline{\xi^2}(t, x) = \int_{\mathbb{R}} (v - u(t, x))^2 M_{t,x}(dv), \quad \overline{\xi^3}(t, x) = \int_{\mathbb{R}} (v - u(t, x))^3 M_{t,x}(dv), \tag{5.15}$$

it follows that

$$\begin{aligned}\overline{v^2}(t, x) &= u^2(t, x) + \overline{\xi^2}(t, x), \\ \overline{v^3}(t, x) &= u^3(t, x) + 3u(t, x)\overline{\xi^2}(t, x) + \overline{\xi^3}(t, x).\end{aligned} \tag{5.16}$$

Then (5.13) for $g(v) = 1, v$, and $\frac{1}{2}v^2$ gives

(5.17)
$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} (\partial_t\phi + \partial_x\phi u)\mu_t(dx)dt + \int_{\mathbb{R}} \phi(0, x)\mu_0(dx) = 0, \\ &\int_0^T \int_{\mathbb{R}} \left(\partial_t\phi u + \partial_x\left(\phi u^2 + \overline{\xi^2}\right)\right) \mu_t(dx)dt + \int_{\mathbb{R}} \phi(0, x)u\, \mu_0(dx) = 0, \\ &\int_0^T \int_{\mathbb{R}} \left\{\partial_t\phi \left(\frac{1}{2}u^2 + \frac{1}{2}\overline{\xi^2}\right) + \partial_x\phi \left[\left(\frac{1}{2}u^2 + \frac{3}{2}\overline{\xi^2}\right) u + \frac{\overline{\xi^3}}{2}\right]\right\} \mu_t(dx)dt \\ &\qquad + \int_{\mathbb{R}} \phi(0, x) \left(\frac{1}{2}u^2(0, x) + \frac{1}{2}\overline{\xi^2}(0, x)\right) \mu_0(dx) = 0.\end{aligned}$$

Moreover, if $\mu_t(dx) = \rho(t,x)dx$, $\overline{\xi^3}(t,x) = 0$ and for $e(t,x) = \dfrac{\overline{\xi^2}(t,x)}{2}$, $p = 2\rho e$, (5.17) shows that ρ, u, e solve weakly the Cauchy problem

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x\left(\rho u^2\right) + \partial_x p = 0 \\ \partial_t\left(\rho \dfrac{u^2}{2} + \rho e\right) + \partial_x\left(\rho u\left(\dfrac{u^2}{2} + e\right) + pu\right) = 0, \\ p = 2\rho e, \\ \rho|_{t=0} = \rho(0,x), \quad u|_{t=0} = u(0,x), \quad e|_{t=0} = e(0,x), \end{cases} \tag{5.18}$$

the one dimensional Euler system, cf. [CF], p. 7. In summary, we have shown:

Proposition 5.4. *For $M_t(dx,dv) = S_t M_0(dx,dv)$, suppose that $v^3 \in L^1(M_0)$, $\mu_t(dx) = \rho(t,x)dx$, and $\overline{\xi^3}(t,x) = 0$. Then $\rho(t,x), u(t,x), e(t,x)$ as defined above is a weak solution to the one dimensional Euler system* (5.18)*.*

The definition of initial conditions for weak solutions here is compatible with the one in [dP], p. 2 and [VF], §VII.10. Two examples satisfying the conditions of this proposition now follow.

5.3. **Two-layer system.** For N fixed, consider $N = 2n$ point molecules x_1, $x_2, \dots, x_N$ on the real line, with

$$x_k(0) = \frac{k}{N}, \quad u_k(0) = \begin{cases} 1 & \text{for } k \text{ odd} \\ -1 & \text{for } k \text{ even.} \end{cases} \tag{5.19}$$

Let the system evolve as in subsection 5.1. After the first n simultaneous collisions take place the molecules with labels 1 and N move with velocities 1 and -1, respectively, without ever interacting with any other molecule again. The remaining molecules now form a replica of the initial system, reduced by two molecules.

As in [Z], the graphs of the positions as functions of time show the evolution of the system, Figure 10. For

$$M_t^{(N)} = \frac{1}{N} \sum_{k=1}^{N} \delta_{(x_k(t), u_k(t))}, \tag{5.20}$$

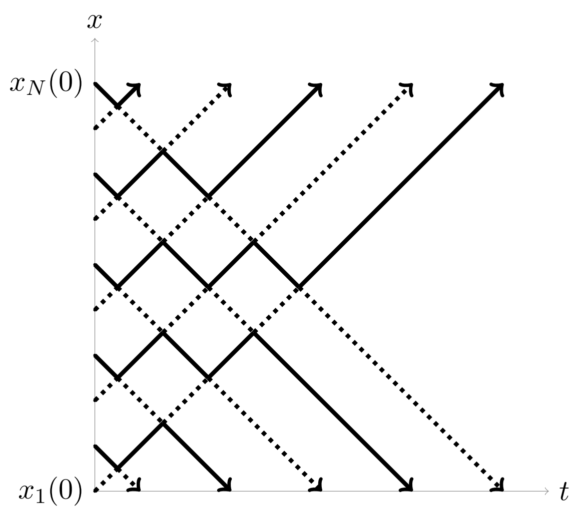

FIGURE 10. Microscopic evolution of subsection 5.3.

according to Proposition 5.1,

$$M_t^{(N)} = S_t M_0^{(N)}. \tag{5.21}$$

On the other hand, it is easy to check that as $N \to \infty$,

$$M_0^{(N)}(dx, dv) \Rightarrow M_0(dx, dv) = \chi_{[0,1]}(x)dx \otimes \left(\frac{1}{2}\delta_{-1}(dv) + \frac{1}{2}\delta_1(dv)\right), \tag{5.22}$$

therefore, by Lemma 5.2,

$$M_t^{(N)} \Rightarrow M_t = S_t M_0, \quad N \to \infty. \tag{5.23}$$

It is straightforward to calculate that

$$M_t(dx, dv) = \frac{1}{2}\chi_{[t,t+1]}(x)dx \otimes \delta_1(dv) + \frac{1}{2}\chi_{[-t,-t+1]}(x)dx \otimes \delta_{-1}(dv). \tag{5.24}$$

M_t describes two layers, each of total mass $1/2$, initially overlapping on the interval $[0, 1]$, moving with velocities ± 1 for $t \geq 0$, see Figure 11. The macroscopic density,

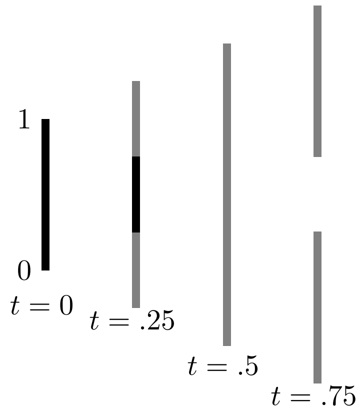

FIGURE 11. Macroscopic evolution of subsection 5.3.

velocity and energy density given by M_t are

$$\begin{cases} \rho(t,x) = \frac{1}{2}\chi_{[t,1+t]}(x) + \frac{1}{2}\chi_{[-t,1-t]}(x) \\ u(t,x) = \chi_{[t,1+t]}(x) - \chi_{[-t,1-t]}(x), \\ e(t,x) = \frac{1}{2}\chi_{[-t,1-t]}(x) \cdot \chi_{[t,1+t]}(x). \end{cases} \tag{5.25}$$

Notice that $\int_{\mathbb{R}^2} |v|^3 M_0(dx,dv) < \infty$ and

$$\overline{\xi^3} = \int_{\mathbb{R}} (v - u(t,x))^3 M_{t,x}(dv) = 0. \tag{5.26}$$

Therefore, by Proposition 5.4, (ρ, u, e) is a solution to the Euler system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x\left(\rho u^2\right) + \partial_x p = 0 \\ \partial_t\left(\rho \frac{u^2}{2} + \rho e\right) + \partial_x\left(\rho u\left(\frac{u^2}{2} + e\right) + pu\right) = 0, \\ p = 2\rho e, \\ \rho|_{t=0} = \chi_{[0,1]}(x), \quad u|_{t=0} = 0, \quad e|_{t=0} = \frac{1}{2}\chi_{[0,1]}(x). \end{cases} \tag{5.27}$$

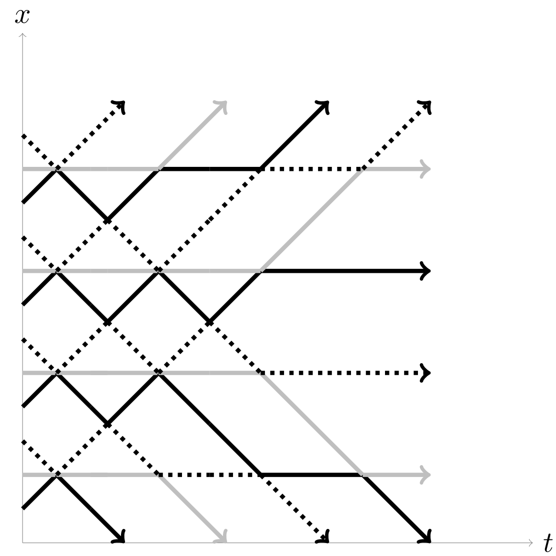

FIGURE 12. Microscopic evolution of subsection 5.4.

5.4. **Three-layer system.** Consider now for each $N = 3n$ a second system, consisting of N molecules $x_1, x_2, \dots, x_N$ on the real line with

$$x_k(0) = \frac{k}{N}, \quad k = 1, \dots, N, \tag{5.28}$$
$$u_k(0) = \begin{cases} \sqrt{6}/2 & \text{for } k = 3m-2 \\ 0 & \text{for } k = 3m-1 \\ -\sqrt{6}/2 & \text{for } k = 3m, \qquad m = 1, \dots, n, \end{cases}$$

also evolving under elastic collisions as in section 5.1.

The evolution of the system initialized by (5.28) is shown in Figure 12. Again, if for the current system

$$\widetilde{M}_t^{(N)} = \frac{1}{N} \sum_{k=1}^{N} \delta_{(x_k(t), u_k(t))}, \tag{5.29}$$

by Proposition 5.1,

$$\widetilde{M}_t^{(N)} = S_t \widetilde{M}_0^{(N)}. \tag{5.30}$$

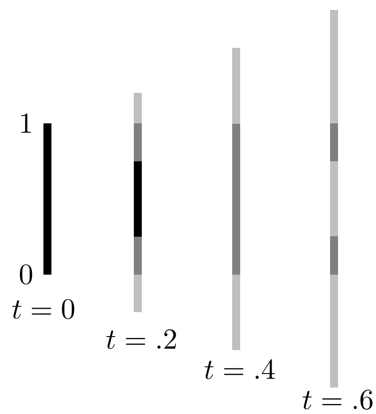

FIGURE 13. Macroscopic evolution of subsection 5.4.

On the other hand, as $N \to \infty$,

$$\begin{aligned}\widetilde{M}_0^{(N)}(dx,dv) &\Rightarrow \widetilde{M}_0(dx,dv) \\ &= \chi_{[0,1]}(x)dx \otimes \left(\frac{1}{3}\delta_{-\sqrt{6}/2}(dv) + \frac{1}{3}\delta_0(dv) + \frac{1}{3}\delta_{\sqrt{6}/2}(dv)\right),\end{aligned} \tag{5.31}$$

therefore

$$\widetilde{M}_t^{(N)} \Rightarrow \widetilde{M}_t = S_t \widetilde{M}_0, \quad N \to \infty. \tag{5.32}$$

It is again a straightforward calculation that

$$\begin{aligned}\widetilde{M}_t(dx,dv) =& \frac{1}{3}\chi_{\left[-\frac{\sqrt{6}}{2}t,\ 1-\frac{\sqrt{6}}{2}t\right]}(x)dx \otimes \delta_{-\frac{\sqrt{6}}{2}}(dv) \\ &+ \frac{1}{3}\chi_{[0,1]}(x)dx \otimes \delta_0(dv) + \frac{1}{3}\chi_{\left[\frac{\sqrt{6}}{2}t,\ 1+\frac{\sqrt{6}}{2}t\right]}(x)dx \otimes \delta_{\frac{\sqrt{6}}{2}}(dv).\end{aligned} \tag{5.33}$$

$\widetilde{M}_t$ describes three layers, each of total mass $1/3$, initially overlapping on the interval $[0,1]$. Two of them move with velocities $\pm\sqrt{6}/2$ for $t > 0$, while the third stays at rest, see Figure 13. The macroscopic density, velocity and energy density

given by $\widetilde{M}_t$ are

$$\left\{\begin{aligned} \widetilde{\rho}(t,x) &= \tfrac{1}{3}\chi_{\left[-\frac{\sqrt{6}}{2}t,\ 1-\frac{\sqrt{6}}{2}t\right]}(x) + \tfrac{1}{3}\chi_{[0,1]}(x) + \tfrac{1}{3}\chi_{\left[\frac{\sqrt{6}}{2}t,\ 1+\frac{\sqrt{6}}{2}t\right]}(x) \\ \widetilde{u}(t,x) &= \frac{-\frac{\sqrt{6}}{6}\chi_{\left[-\frac{\sqrt{6}}{2}t,\ 1-\frac{\sqrt{6}}{2}t\right]}(x) + \frac{\sqrt{6}}{6}\chi_{\left[\frac{\sqrt{6}}{2}t,\ 1+\frac{\sqrt{6}}{2}t\right]}(x)}{\widetilde{\rho}(t,x)} \\ \widetilde{e}(t,x) &= \frac{\frac{1}{4}\chi_{\left[-\frac{\sqrt{6}}{2}t,\ 1-\frac{\sqrt{6}}{2}t\right]}(x) + \frac{1}{4}\chi_{\left[\frac{\sqrt{6}}{2}t,\ 1+\frac{\sqrt{6}}{2}t\right]}(x) - \frac{1}{2}\widetilde{\rho}(t,x)\widetilde{u}^2(t,x)}{\widetilde{\rho}(t,x)}. \end{aligned}\right. \tag{5.34}$$

When $\widetilde{\rho}(t,x) = 0$, take $\widetilde{u}(t,x), \widetilde{e}(t,x) = 0$. Notice that

$$\overline{\xi^3}^{\sim}(0,x) = \int_{\mathbb{R}} (v - \widetilde{u}(t,x))^3 \widetilde{M}_{t,x}(dv) = 0. \tag{5.35}$$

By Proposition 5.4, $(\widetilde{\rho}, \widetilde{u}, \widetilde{e})$ is also a solution to the Cauchy problem (5.27), clearly distinct from the solution (ρ, u, e).

Remark 5.5. *It is well known that weak solutions to systems like* (5.27) *are not unique, see* [D]. *This section provides a microscopic interpretation of such macroscopic non-uniqueness, showing that such phenomena are quite natural from a Hamiltonian point of view.*

Appendix : Motion in a Central Field

We establish some facts for the motion in dimension 2 of a single particle in an external field of potential energy Φ of finite range σ:

$$\mathbf{x}''(t) = -\Phi'(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}. \tag{A.1}$$

To accommodate (2.2), assume that $\Phi : (0,\infty) \to [0,\infty)$ satisfies

$$\lim_{r\to 0} \Phi(r) = +\infty, \quad \Phi' \le 0, \quad \Phi'' \ge 0, \quad \Phi(r) \neq 0 \Leftrightarrow 0 < r < \sigma. \tag{A.2}$$

Consulting Figure 14, let O be the center of the potential Φ. A molecule m enters the range of Φ at A with velocity $\mathbf{v}$ and leaves at B. For D the middle of AB, the path of m in the range of Φ is symmetric about OD, by the reversibility of the equations of motion. Decompose $\mathbf{v}(t)$ into $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$ along AB and OD, respectively, and let E be the intersection of OD and the trajectory of m. When

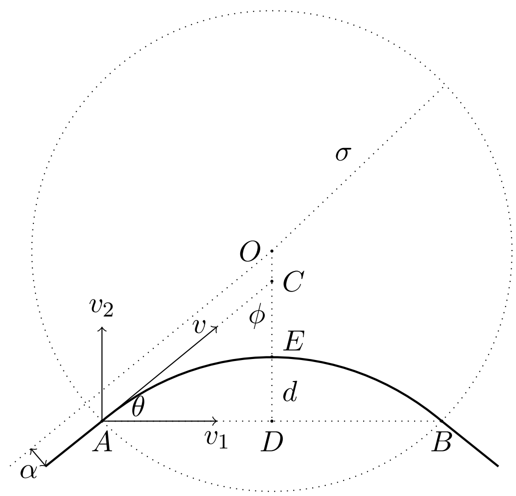

FIGURE 14. Motion in a central field of finite range σ.

m crosses OD it has moved d on the direction of OD. If θ is the angle between $\mathbf{v}$ and AB and C is the point on OD with AC of direction $\mathbf{v}$, then

$$d = DE < CD = AC \cdot \sin\theta < AO \cdot \sin\theta = \sigma \sin\theta. \tag{A.3}$$

Let T be the time it takes m to travel from A to B.

Lemma A.1. *For σ the range of Φ, T and v as above satisfy $T < \dfrac{4\sigma}{v}$.*

Proof. From (A.1),

$$v_2'' = -\Phi''(|\mathbf{x}|)\frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^2}x_2 - \Phi'(|\mathbf{x}|)\frac{x_2'}{|\mathbf{x}|} + \Phi'(|\mathbf{x}|)\frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3}x_2. \tag{A.4}$$

For $x_2 < 0$ and as $\dfrac{d|\mathbf{x}|^2}{dt} < 0$ for $t \in (0, T/2)$, and as Φ is convex, the first term of this is negative and, if x_1 is also negative, the sum of the remaining two terms is also negative provided that

$$-x_2'|\mathbf{x}|^2 + x_2(\mathbf{x}\cdot\mathbf{x}') > 0 \Leftrightarrow -x_2'x_1 + x_2x_1' < 0, \tag{A.5}$$

since Φ is decreasing. Now note that $-x_2'x_1 + x_2x_1'$ stays constant in time and the inequality is satified at $t = 0$. Therefore v_2 is concave and by (A.3)

$$\frac{v_2(0)}{2} \cdot \frac{T}{2} < d < \sigma \sin\theta, \tag{A.6}$$

which, along with $v_2(0) = v \sin\theta$, concludes the proof. □

Still in Figure 14, let $\angle ACD = \phi$. Denoting the distance of O from AC (the impact parameter) by α, by [LL], p. 49[1]

$$\phi(\alpha) = \int_{r_{min}}^{\infty} \frac{\alpha}{r^2\sqrt{1 - \frac{\alpha^2}{r^2} - \frac{\Phi(r)}{E}}} dr, \tag{A.7}$$

where $E = \frac{1}{2}mv^2$ and r_{min} is a zero of the radicand:

$$1 - \frac{\alpha^2}{r_{min}^2} - \frac{\Phi(r_{min})}{E} = 0. \tag{A.8}$$

Lemma A.2. *For interaction potential as in* (2.2)*,* $r_{min} = r_{min}(\alpha)$ *is increasing and* $\phi(\alpha)$ *is continuous on* $[0, \infty)$*.*

Proof. For fixed α and E, the function

$$r \mapsto \frac{\alpha^2}{r^2} + \frac{\Phi(r)}{E} \tag{A.9}$$

is strictly decreasing from $+\infty$ to 0 for $r > 0$ and the pre-image r_{min} of 1 satisfies (A.8), or

$$\alpha = \left(1 - \frac{\Phi(r_{min})}{E}\right)^{1/2} r_{min} \tag{A.10}$$

showing that $\alpha = \alpha(r_{min})$, and therefore $r_{min} = r_{min}(\alpha)$, is increasing.

[1]Note here that [LL]'s analysis of motion in a central field in their §14 is valid for any central field, including the ones with finite range.

To show that ϕ is continuous, change the variable in (A.7) via $r = r_{min}y$:

$$\begin{aligned}\phi(\alpha) &= \int_1^\infty \frac{1}{y^2\sqrt{\frac{r_{min}^2}{\alpha^2}\left(1-\frac{\Phi(r_{min}y)}{E}\right)-\frac{1}{y^2}}}dy \\ &\overset{\text{by (A.10)}}{=} \int_1^\infty \frac{1}{y^2\sqrt{\frac{E-\Phi(r_{min}y)}{E-\Phi(r_{min})}-\frac{1}{y^2}}}dy.\end{aligned} \tag{A.11}$$

From (A.8) we have

$$E > \Phi(r_{min}) \tag{A.12}$$

and since $\Phi(r)$ is decreasing,

$$\Phi(r_{min}) \geq \Phi(r_{min}y), \quad y \geq 1, \tag{A.13}$$

therefore

$$\frac{1}{y^2\sqrt{\frac{E-\Phi(r_{min}y)}{E-\Phi(r_{min})}-\frac{1}{y^2}}} \leq \frac{1}{y^2\sqrt{1-\frac{1}{y^2}}}, \tag{A.14}$$

with

$$\int_1^\infty \frac{1}{y^2\sqrt{1-\frac{1}{y^2}}}dy = \frac{\pi}{2}. \tag{A.15}$$

In other words, the integrand of ϕ is dominated by an integrable function. This, and the continuity of r_{min} in α, show that ϕ is continuous in α. □

Corollary A.3. *For any $0 \leq \phi_0 \leq \pi/2$, there exists $0 \leq \alpha_0 \leq \sigma$ such that $\phi(\alpha_0) = \phi_0$.*

Proof. Just use continuity and that $\phi(0) = 0$ ("head-on collision"), $\phi(\sigma) = \frac{\pi}{2}$ (no interaction). □

As is well known, motion in a central field also describes a system of two bodies interacting with each other via Φ, a function of their distance, in a coordinate

system with its origin at the center of mass of the system. The formulas for this transformation are in [LL], §13.

References

[AGS] Ambrosio L., N. Gigli, G. Savaré, *Gradient Flows*, Birkäuser, 2005

[B] Boltzmann, L., *Lectures on gas theory*, University of California Press, 1964

[BN] Bressan, A., T. Nguyen, *Non-existence and Non-uniqueness for Multidimensional Sticky Particle Systems*, Kinetic and Related Models, **7** (2) (2014), 205–218.

[CF] Courant, R., K.O. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience, 1948

[D] Dafermos, C.M., *Hyperbolic conservation laws in continuum physics*, vol. 325 of Grundlehren der Mathematischen Wissenschaften, Springer, 2000.

[dLS] De Lellis, C., L. Székelyhidi, *On admissibility criteria for weak solutions of the Euler equations*, Arch. Ration. Mech. Anal. **195** (1) (2010), 225–260

[dP] DiPerna, R.J., *Convergence of the viscosity method for isentropic gas dynamics*, Comm. Math. Phys. **91** (1) (1983), 1–30

[DJX] Dostoglou, S., N.C. Jacob, Jianfei Xue, *On hydrodynamic equations at the limit of infinitely many molecules*, J. Mathematical Sciences, New York **205** (2) (2015), 91–104

[ERS] E, Weinan, Yu.G. Rykov, Ya.G. Sinai, *Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics* Comm. Math. Phys. Vol. **177**, (2) (1996), 349-380.

[G] Gal'perin, G.A., *On Systems of Locally Interacting and Repelling Particles Moving in Space*, Trudy Moskov. Mat. Obshch. **43** (1981), 142–196

[L] Lanford, O.E., III, *Time evolution of large classical systems*, Lecture Notes in Phys. **38**, Dynamical systems, theory and applications (Recontres, Battelle Res. Inst., Seattle, Wash., 1974), Springer, Berlin, 1975, 1–111

[LL] Landau, L.D., E.M. Lifshitz, *Mechanics*, 3rd Edition, Vol. 1 of Course of Theoretical Physics, Pergamon Press, 1976.

[M] Maxwell, J.C., *On the Dynamical Theory of Gases*, Philosophical Transactions of the Royal Society of London **157** (1867), 49–88.

[Mor] Morrey, C.B., *On the derivation of the Equations of Hydrodynamics from Statistical Mechanics*, Comm. Pure and Applied Math. **VIII** (1955), 317–322.

[N] Navier, C.L., *Sur Le Lois du Movement des Fluides*, Mémoires, Academie des Sciences **6** (1827), 389–440 (read 1822)

[Sch] Scheffer, V., *An Inviscid Flow with Compact Support in Space-Time*, J. Geometric Analysis **3** (4) (1993), 343–401.

[Sh] Shnirelman, A., *On the Nonuniqueness of Weak Solution of the Euelr Equation*, Comm. Pure Applied Math. **L** (1997), 1261–1286

[V] Vaserstein, L.N., *On Systems of Particles with Finite-Range and/or Repulsive Interactions*, Commun. Math. Phys. **69** (1979), 31–56

[VF] Vishik, M.I., A.V. Fursikov, *Mathematical Problems of Statistical Hydromechanics*, Kluwer 1988

[X] Xue, Jianfei, University of Missouri Ph.D. Thesis, in preparation.

[Z] Zemlyakov, A.N., *The arithmetic and geometry of collisions*, Kvant **4**, (1978), 14–21.

Department of Mathematics, University of Missouri, Columbia, MO 65211